ON THE EULER NUMBERS AND ITS APPLICATIONS

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Abstract. Recently, the q-Euler numbers and polynomials are constructed in [T. Kim, The modified q-Euler numbers and polynomials, Advanced Studies in Contemporary Mathematics, 16(2008), 161-170]. These q-Euler numbers and polynomials contain the interesting properties. In this paper we prove Von-Staudt Clausen's type theorem related to the q-Euler numbers. That is, we prove that the q-Euler numbers are p-adic integers. Finally, we give the proof of Kummer type congruences for the q-Euler numbers.

§1. Introduction/Definition

Let p be a fixed odd prime. Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$, and $\mathbb{C}_p$ will, respectively, denote the ring of p-adic rational integers, the field of p-adic rational numbers, the complex number field, and the completion of algebraic closure of $\mathbb{Q}_p$. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = \frac{1}{p}$. When one talks of q-extension, q is variously considered as an indeterminate, a complex $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$, see [9-22]. If $q \in \mathbb{C}$, then we assume $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume $|1-q|_p < p^{-\frac{1}{p-1}}$. The ordinary Euler numbers are defined as

$$\frac{2}{e^t+1} = e^{Et} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!},$$

where we use the technique method notation by replacing E^n by E^n $(n \geq 0)$, symbolically (see [1-23]). From this definition, we can derive the following relation.

$$E_0 = 1, \text{ and } (E+1)^n + E_n = 2\delta_{0,n}, \text{ where } \delta_{0,n} \text{ is Kronecker symbol.}$$

For $x \in \mathbb{Q}_p$ (or $\mathbb{R}$), we use the notation $[x]_q = \frac{1-q^x}{1-q}$, and $[x]_{-q} = \frac{1-(-q)^x}{1+q}$, see [5-6]. In [5], the q-Euler numbers are defined as

$$E_{0,q} = \frac{[2]_q}{2}, \text{ and } (qE+1)^n + E_{n,q} = [2]_q \delta_{0,n}, \tag{1}$$

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with the usual convention of replacing E^n by $E_{n,q}$. Note that $\lim_{q\to 1} E_{n,q} = E_n$. For a fixed positive integer d with $(p,d)=1$, let

$$X = X_d = \varprojlim_N \mathbb{Z}/dp^N\mathbb{Z}, \quad X_1 = \mathbb{Z}_p,$$

$$X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} a + dp\mathbb{Z}_p,$$

$$a + dp^N\mathbb{Z}_p = \{x \in X \mid x \equiv a \pmod{dp^N}\},$$

where $a \in \mathbb{Z}$ lies in $0 \leq a < dp^N$, (see [4-23]). We say that f is a uniformly differentiable function at a point $a \in \mathbb{Z}_p$ and denote this property by $f \in UD(\mathbb{Z}_p)$, if the difference quotients $F_f(x,y) = \dfrac{f(x)-f(y)}{x-y}$ have a limit $l = f'(a)$ as $(x,y) \to (a,a)$. For $f \in UD(\mathbb{Z}_p)$, let us start with the expression

$$\frac{1}{[p^N]_q} \sum_{0 \leq j < p^N} q^j f(j) = \sum_{0 \leq j < p^N} f(j)\mu_q(j + p^N\mathbb{Z}_p),$$

representing a q-analogue of Riemann sums for f, see [5, 6]. The integral of f on $\mathbb{Z}_p$ will be defined as limit $(n \to \infty)$ of those sums, when it exists. The q-deformed bosonic p-adic integral of the function $f \in UD(\mathbb{Z}_p)$ is defined by

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[dp^N]_q} \sum_{0 \leq x < dp^N} f(x) q^x, \text{ see [5]}. \tag{2}$$

In the sense of fermionic, let us define the fermionic p-adic q-integral as

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x, \text{ see [5-10]}. \tag{3}$$

From (3) we note that

$$qI_{-q}(f_1) + I_{-q}(f) = [2]_q f(0), \text{ where } f_1(x) = f(x+1). \tag{4}$$

In [5], the Witt's type formula for the q-Euler numbers $E_{n,q}$ id given by

$$\sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} [x]_q^n q^{-x} d\mu_{-q}(x) \frac{t^n}{n!} = \int_{\mathbb{Z}_p} e^{[x]_q t} q^{-x} d\mu_{-q}(x) = \sum_{n=0}^{\infty} E_{n,q} \frac{t^n}{n!}. \tag{5}$$

By comparing the coefficients on both sides in (5), we see that

$$\int_{\mathbb{Z}_p} [x]_q^n q^{-x} d\mu_{-q}(x) = E_{n,q}, \text{ see [5]}. \tag{6}$$

By the definition of the fermionic p-adic q-integral on $\mathbb{Z}_p$, the q-Euler polynomials are also defined as

$$(7)\qquad \int_{\mathbb{Z}_p} e^{[x+y]_q t} q^{-y} d\mu_{-q}(y) = e^{[x]_q t} \int_{\mathbb{Z}_p} e^{q^x [y]_q t} q^{-y} d\mu_{-q}(y) = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!}.$$

From (6) and (7), we note that

$$(8)\qquad E_{n,q}(x) = \sum_{k=0}^{n} \binom{n}{k} [x]_q^{n-k} q^k E_{k,q}, \text{ where } \binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}.$$

Let $F_q(t,x)$ be the generating function of the q-Euler polynomials. Then we have

$$(9)\qquad F_q(t,x) = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!} = [2]_q \sum_{k=0}^{\infty} (-1)^k e^{[k+x]_q t}.$$

Let χ be the Dirichlet's character with odd conductor $d \in \mathbb{N}$. Then the generalized q-Euler numbers attached to χ are defined as

$$(10)\qquad E_{n,\chi,q} = \int_X [x]_q^n q^{-x} \chi(x) d\mu_{-q}(x) = [d]_q^n \frac{[2]_q}{[2]_{q^d}} \sum_{a=0}^{d-1} \chi(a)(-1)^a E_{n,q^d}(\frac{a}{d}).$$

Let $F_{\chi,q}(t) = \sum_{n=0}^{\infty} E_{n,\chi,q} \frac{t^n}{n!}$. Then we note that

$$(11)\qquad F_{\chi,q}(t) = \sum_{n=0}^{\infty} E_{n,\chi,q} \frac{t^n}{n!} = [2]_q \sum_{n=0}^{\infty} \chi(n)(-1)^n e^{[n]_q t}.$$

In this paper we prove the Von-Staudt-Clausen's type theorem related to the q-Euler numbers. That is, we prove that the q-Euler numbers are the p-adic integers. Finally, we give the proofs of the Kummer congruences for the q-Euler numbers.

§2. q-Euler numbers and polynomials

From (1)and (6) we derive

$$E_{n,q} = \int_{\mathbb{Z}_p} q^{-x} [x]_q^n d\mu_{-q}(x) = \frac{[2]_q}{2} \int_{\mathbb{Z}_p} [x]_q^n d\mu_{-1}(x).$$

Thus, we note that $\lim_{n\to\infty} E_{n,q} = E_n = \int_{\mathbb{Z}_p} x^n d\mu_{-1}$. For $q \in \mathbb{C}_p$ with $|1-q|_p < p^{-\frac{1}{p-1}}$, we have

$$(-1)^j [j]_q - j(-1)^j = (-1)^j \left(\frac{\sum_{l=0}^{j} \binom{j}{l} (q-1)^l - 1}{q-1} - j \right) = (-1)^j \sum_{l=2}^{j} \binom{j}{l} (q-1)^{l-1}.$$

Thus, we see that

$$\left|(-1)^j([j]_q - j)\right|_p \le \max_{2\le l\le j}\left(|(q-1)|_p^{l-1}\right) = |q-1|_p. \tag{12}$$

From (12), we can derive

$$\begin{aligned}\left|\sum_{j=0}^{p-1}(-1)^j[j]_q\right|_p &= \left|\sum_{j=0}^{p-1}(-1)^j([j]_q - j) + \sum_{j=0}^{p-1}(-1)^j j\right|_p \\ &= \left|\sum_{j=0}^{p-1}(-1)^j([j]_q - j) + \frac{p-1}{2}\right|_p \le 1.\end{aligned} \tag{13}$$

For $k \ge 1$, let

$$T_n(k) = \sum_{x=0}^{p^k-1}(-1)^x[x]_q^n = [0]_q^n - [1]_q^n + \cdots + [p^k-1]_q^n. \tag{14}$$

Note that $\lim_{k\to\infty} T_n(k) = \frac{2}{[2]_q}E_{n,q}$. From (14), we can derive

(15)

$$\begin{aligned} T_n(k+1) &= \sum_{x=0}^{p^{k+1}-1}(-1)^x[x]_q^n = \sum_{i=0}^{p^k-1}\sum_{j=0}^{p-1}[i+jp^k]_q^n(-1)^{i+jp^k} \\ &= \sum_{i=0}^{p^k-1}\sum_{j=0}^{p-1}\left([i]_q + q^i[jp^k]_q\right)^n(-1)^{i+jp^k} = \sum_{i=0}^{p^k-1}\sum_{j=0}^{p-1}\sum_{l=0}^{n}\binom{n}{l}[i]_q^{n-l}q^{il}[jp^k]_q^l(-1)^{i+jp^k} \\ &= \sum_{i=0}^{p^k-1}\sum_{j=0}^{p-1}\sum_{l=0}^{n}\binom{n}{l}[i]_q^{n-l}q^{il}[p^k]_q^l[j]_{q^{p^k}}^l(-1)^{i+j} \\ &= \sum_{i=0}^{p^k-1}[i]_q^n(-1)^i + \sum_{i=0}^{p^k-1}\sum_{j=0}^{p-1}\sum_{l=1}^{n}\binom{n}{l}[i]_q^{n-l}q^{il}[p^k]_q^l[j]_{q^{p^k}}^l. \end{aligned}$$

Thus, we have

$$T_n(k+1) \equiv \sum_{i=0}^{p^k-1}[i]_q^n(-1)^i \pmod{[p^k]_q}. \tag{16}$$

From (15) we note that

(17)

$$\begin{aligned}
\sum_{x=0}^{p^{k+1}-1}[x]_q^n(-1)^x &= \sum_{j=0}^{p-1}\sum_{i=0}^{p^k-1}[j+ip]_q^n(-1)^{j+ip} \\
&= \sum_{j=0}^{p-1}(-1)^j\sum_{i=0}^{p^k-1}(-1)^i\sum_{l=0}^{n}\binom{n}{l}[j]_q^{n-l}q^{jl}[p]_q^l[i]_{q^p}^l \\
&= \sum_{j=0}^{p-1}(-1)^j[j]_q^n+\sum_{j=0}^{p-1}(-1)^j\sum_{i=0}^{p^k-1}(-1)^i\sum_{l=1}^{n}\binom{n}{l}[j]_q^{n-l}q^{jl}[p]_q^l[i]_{q^p}^l \\
&\equiv \sum_{j=0}^{p-1}(-1)^j[j]_q^n \pmod{[p]_q}.
\end{aligned}$$

By (17), we obtain

$$\sum_{x=0}^{p^{k+1}-1}(-1)^x[x]_q^n \equiv \sum_{j=0}^{p-1}(-1)^j[j]_q^n \pmod{[p]_q}. \tag{18}$$

From (16) and (19), we can also derive

$$\sum_{j=0}^{p-1}(-1)^j[j]_q^n \equiv \frac{2}{[2]_q}\int_X[x]_q^nq^{-x}d\mu_{-q}(x)=\frac{2}{[2]_q}E_{n,q} \pmod{[p]_q}. \tag{19}$$

Thus, we note that

$$\sum_{j=0}^{p-1}(-1)^j[j]_q^n \equiv \frac{2}{[2]_q}E_{n,q} \pmod{[p]_q}. \tag{20}$$

Therefore we obtain the following theorem.

Theorem 1. *For $n \geq 0$, we have*

$$\sum_{j=0}^{p-1}(-1)^j[j]_q^n \equiv \frac{2}{[2]_q}E_{n,q} \pmod{[p]_q}.$$

By (15), (16) and (20), we obtain the following corollary.

Corollary 2. *For $n \geq 0$, we have*

$$\frac{2}{[2]_q}E_{n,q} + \sum_{j=0}^{p-1}(-1)^{j+1}[j]_q^n \in \mathbb{Z}_p.$$

For $n \geq 0$, we note that

$$\begin{aligned}\left|\frac{2}{[2]_q}E_{n,q}\right|_p &= \left|\frac{2}{[2]_q}E_{n,q} - \sum_{j=0}^{p-1}(-1)^j[j]_q^n + \sum_{j=1}^{p-1}(-1)^j[j]_q^n\right|_p \\ &\leq \max\left(\left|\frac{2}{[2]_q}E_{n,q} - \sum_{j=0}^{p-1}(-1)^j[j]_q^n\right|_p, \left|\sum_{j=1}^{p-1}(-1)^j[j]_q^n\right|_p\right).\end{aligned}$$

By (13) and Corollary 2, we obtain the following corollary.

Corollary 3. *For $n \geq 0$, we have*

$$\frac{2}{[2]_q}E_{n,q} \in \mathbb{Z}_p.$$

Let χ be the Dirichlet's character with odd conductor $d(\in \mathbb{N})$. Then the generalized q-Euler numbers attached to χ as follows.

$$\sum_{n=0}^{\infty} E_{n,\chi,q}\frac{t^n}{n!} = [2]_q \sum_{n=0}^{\infty}\chi(n)(-1)^n e^{[n]_q t} = \int_X \chi(x)e^{[x]_q t}q^{-x}d\mu_{-q}(x). \tag{21}$$

We denote $\bar{d} = [d, p]$ the least common multiple of the conductor d of χ and p. From (21), we derive

$$\frac{2}{[2]_q}E_{n,\chi,q} = \frac{2}{[2]_q}\int_X [x]_q^n q^{-x}\chi(x)d\mu_{-q}(x) = \lim_{N\to\infty}\sum_{x=0}^{dp^N-1}[x]_q^n\chi(x)(-1)^x. \tag{22}$$

By (22), we see that

$$\begin{aligned}\frac{2}{[2]_q}E_{n,\chi,q} &= \lim_{\rho\to\infty}\sum_{\substack{1\leq x\leq \bar{d}p^\rho \\ (x,p)=1}}\chi(x)(-1)^x[x]_q^n + \lim_{\rho\to\infty}\sum_{y=1}^{\bar{d}p^{\rho-1}}\chi(p)\chi(y)[p]_q^n[y]_{q^p}^n(-1)^y \\ &= \lim_{\rho\to\infty}\sum_{\substack{1\leq x\leq \bar{d}p^\rho \\ (x,p)=1}}\chi(x)(-1)^x[x]_q^n + \chi(p)[p]_q^n\lim_{\rho\to\infty}\sum_{y=1}^{\bar{d}p^{\rho-1}}\chi(y)[y]_{q^p}^n(-1)^y.\end{aligned}$$

Thus, we have

$$\frac{2}{[2]_q}E_{n,\chi,q}-\chi(p)[p]_q^n\frac{2}{[2]_{q^p}}E_{n,\chi,q^p}=\lim_{\rho\to\infty}\sum_{\substack{1\le x\le \bar{d}p^\rho\\(x,p)=1}}\chi(x)(-1)^x[x]_q^n. \tag{23}$$

Let w denote the Teichmüller character $\mod p$. For $x\in X^*$, we set $<x>=<x:q>=\frac{[x]_q}{w(x)}$. Note that $|<x>-1|_p<p^{-\frac{1}{p-1}}$, where $<x>^s=\exp(s\log_p<x>)$ for $s\in\mathbb{Z}_p$. For $s\in\mathbb{Z}_p$, we define the p-adic q-L-function related to $E_{n,\chi,q}$ as follows.

$$L_{p,q,E}(s,\chi)=\lim_{\rho\to\infty}\sum_{\substack{1\le x\le \bar{d}p^\rho\\(x,p)=1}}\chi(x)(-1)^x<x>^{-s}. \tag{24}$$

For $k\ge 0$, we have

$$\begin{aligned}
L_{p,q,E}(-k,\chi w^k)&=\lim_{\rho\to\infty}\sum_{\substack{1\le x\le \bar{d}p^\rho\\(x,p)=1}}\chi(x)(-1)^x[x]_q^k\\
&=\frac{2}{[2]_q}\int_X[x]_q^k\chi(x)q^{-x}d\mu_{-q}(x)-\frac{2}{[2]_{q^p}}\int_{pX}[x]_q^k\chi(x)q^{-x}d\mu_{-q}(x)\\
&=\frac{2}{[2]_q}\int_X[x]_q^k\chi(x)q^{-x}d\mu_{-q}(x)-\chi(p)[p]_q^k\frac{2}{[2]_{q^p}}\int_X[x]_{q^p}^k\chi(x)q^{-px}d\mu_{-q^p}(x)\\
&=\frac{2}{[2]_q}E_{n,\chi,q}-\chi(p)[p]_q^k\frac{2}{[2]_{q^p}}E_{n,\chi,q^p}.
\end{aligned} \tag{25}$$

It is easy to see that $<x>^{p^n}\equiv 1\ (\mod p^n)$. From the definition of $L_{p,q,E}(s,\chi)$, we can derive

$$\begin{aligned}
L_{p,q,E}(-k,\chi)&=\lim_{\rho\to\infty}\sum_{\substack{1\le x\le \bar{d}p^\rho\\(x,p)=1}}\chi(x)(-1)^x<x>^k\\
&\equiv\lim_{\rho\to\infty}\sum_{\substack{1\le x\le \bar{d}p^\rho\\(x,p)=1}}\chi(x)(-1)^x<x>^{k'}\quad(\mod p^n),
\end{aligned}$$

whenever $k\equiv k'\ (\mod p^n(p-1))$. That is, $L_{p,q,E}(-k,\chi w^k)\equiv L_{p,q,E}(-k',\chi w')\ (\mod p^n)$.

Therefore we obtain the following theorem.

Theorem 4. *(Kummer Congruence) For $k\equiv k'\ (\mod p^n(p-1))$, we have*

$$\frac{2}{[2]_q}E_{k,\chi,q}-\frac{2}{[2]_{q^p}}E_{k,\chi,q^p}\equiv\frac{2}{[2]_q}E_{k',\chi,q}-\frac{2}{[2]_{q^p}}E_{k',\chi,q^p}\ (\mod p^n).$$

Let χ be the primitive Dirichlet's character with conductor p. Then we have

$$\begin{aligned}
&\sum_{x=0}^{p^{N+1}-1}\chi(x)(-1)^x[x]_q^n=\sum_{a=0}^{p-1}\sum_{x=0}^{p^N-1}\chi(a+px)(-1)^{a+px}[a+px]_q^n\\
&=\sum_{a=0}^{p-1}\chi(a)(-1)^a\sum_{x=0}^{p^N-1}(-1)^x([a]_q+q^a[p]_q[x]_{q^p})^n\\
&=\sum_{a=0}^{p-1}\chi(a)(-1)^a\sum_{x=0}^{p^N-1}(-1)^x\sum_{l=0}^{n}\binom{n}{l}[a]_q^{n-l}q^{al}[p]_q^l[x]_{q^p}^l\\
&\equiv\sum_{a=0}^{p-1}\chi(a)(-1)^a[a]_q^n \pmod{[p]_q}.
\end{aligned}$$

If $\rho\to\infty$, then we have

$$\frac{2}{[2]_q}\int_X\chi(x)(-1)^x[x]_q^nq^{-x}d\mu_{-q}(x)\equiv\sum_{a=0}^{p-1}\chi(a)(-1)^a[a]_q^n \pmod{[p]_q}.$$

Thus, we can obtain the following. Let χ be the primitive Dirichlet's character with conductor p. Then we have

$$\frac{2}{[2]_q}E_{n,\chi,q}\equiv\sum_{a=0}^{p-1}\chi(a)(-1)^a[a]_q^n \pmod{[p]_q}. \tag{26}$$

The Eq.(26) also seems to be the new interesting formula. As $q\to 1$, we can also obtain

$$E_{n,\chi}\equiv\sum_{a=0}^{p-1}\chi(a)(-1)^aa^n \pmod{p}.$$

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